International Journal of Theoretical & Applied Sciences, 10(1): 269-276(2018)

ISSN No. (Print): 0975-1718
ISSN No. (Online): 2249-3247

Fixed Point Results in Fuzzy F-Menger Space

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(Received 12 February 2018, Accepted 22 April, 2018)
(Published by Research Trend, Website: www.researchtrend.net)

ABSTRACT: In this present paper we have proved some fixed point results in generating Polish space (random space which is more general than the other spaces) with implicit relation, notable completeness of the space is not a compulsion.

Keyword: Fixed point, Random space, Polish spaces, $JSR - mapping$, Implicit relation.
Mathematics Subject Classification: 47H10, 54H25.

I. INTRODUCTION

The notion of probabilistic metric space was introduced by Menger in 1942 [13] and the first result about the existence of a fixed point of a mapping which is defined on a Menger space is obtained by Sehgel and Barucha-Reid [18].

A number of fixed point theorems for single valued and multi-valued mappings in Menger probabilistic metric space have been considered by many authors [1], [2],[4],[5],[6],[8],[9],[17],[23]. In 1998, Jungck and Rhodes [10] introduced the concept weakly compatible maps and proved many theorems in metric space. Hybrid fixed point theory for nonlinear single-valued and multi-valued maps is a new development in the domain of contraction type multi-valued theory discussed by [3], [7], [15],[16],[19] and [22].

Jungck and Rhoades [10] introduced the weak compatibility to the setting of single valued and multi-valued maps. Singh and Mishra [21] introduced (IT)-commutativity for hybrid pair of single valued and multi-valued maps which need not be weakly compatible. In 2005, Mihet [14] proved a fixed point theorem concerning probabilistic contractions satisfying an implicit relation. Shrivastav *et al.* [20] and others [11], [12], proved fixed point result in fuzzy probabilistic metric space.

In this chapter, we choose to utilize the notion of occasionally weakly compatibility to prove our results in fuzzy F-Menger space, which is a wider and suitable framework in many situations and use point wise R-weakly commuting mappings in fuzzy probabilistic metric spaces satisfying contractive type implicit relations. Here one may observe that we need not impose the completeness requirement of the space or the containment of the ranges of the involved mappings.

II. PRELIMINARIES

Let us define and recall some definitions**:**

Definition 2.1: A fuzzy Fprobabilistic metric space (F FPM space) is an ordered pair $(X,\ F^2_\alpha)$ consisting of a nonempty set X and a mapping F^2_α from $X \times X$ into the collections of all distribution functions $F^2_\alpha \in R \times R$for all $\alpha \in [0,1]$. For $x, y \in X$ we denote the distribution function $F^2_\alpha(x, y)$ by $F^2_{\alpha(x,y)}$and $F^2_{\alpha(x,y)}(u)$ is the value of $F^2_{\alpha(x,y)}$at u in R. The functions $F^2_{\alpha(x,y)}$for all $\alpha \in [0,1]$ assumed to satisfy the following conditions:

(a) $\boldsymbol{F^2}_{\alpha(x,y)}(\boldsymbol{u}) = \mathbf{1} \forall\ \boldsymbol{u}\ >\ 0\ iff\ x\ =\ y,$

(b) $\boldsymbol{F^2}_{\alpha(x,y)}(\mathbf{0}) =\ \mathbf{0}\ \ \forall\ \boldsymbol{x}\,,\boldsymbol{y}\ \boldsymbol{in}\ \boldsymbol{X},$

(c) $\boldsymbol{F^2}_{\alpha(x,y)} = \boldsymbol{F^2}_{\alpha(y,x)} \forall\ \boldsymbol{x}\,,\boldsymbol{y}\ \boldsymbol{in}\ \boldsymbol{X},$

(d) $\boldsymbol{If}\ \boldsymbol{F^2}_{\alpha(x,y)}(\boldsymbol{u}) =\ \mathbf{1}$ and $\boldsymbol{F^2}_{\alpha(y,z)}(\boldsymbol{v}) =\ \mathbf{1}$

$$\Rightarrow F^2{}_{\alpha(x,z)}(u+v) =\ 1\ \ \forall\, x\,, y\,, z\ \in X\ and\ u, v >\ 0.$$

Definition 2.2: A commutative, associative and non-decreasing mapping $t{:}\,[0,1] \times [0,1] \to\ [0,1]$ is a t-norm if and only if $t(a, 1) = 0$ for all $a \in [0,1], t(0,0)\ =\ 0$and $t(c, d) \geq\ t(a, b)$for $\ c \geq a, d \geq\ b.$

Definition 2.3: A Fuzzy F-Menger space is a triplet (X, F_α^2, t), where (X, F_α^2) is a FPM-space,t is a t-norm and the generalized triangle inequality

$$F^2{}_{\alpha(x,z)}(u+v) \geq t\,(F^2{}_{\alpha(x,z)}(u), F^2{}_{\alpha(y,z)}(v))$$

Holds for all $x, y, z\ in\ X\ \ u, v > 0$ and $\alpha \in [0,1]$.

The concept of neighborhoods in Fuzzy **F**-Menger space is introduced as

Definition 2.4: Let (X, F_α^2, t)be a Fuzzy F-Menger space. If $x \in X$, $\varepsilon > 0$ and $\lambda \in (0, 1)$, then (ε, λ)- neighborhood of x, called $U_x(\varepsilon, \lambda)$ is defined by

$$U_x(\varepsilon, \lambda) = \{y \in X: F_{\alpha(x,y)}^2(\varepsilon) > (1-\lambda)\}$$

An(ε, λ)- topology in X is the topology induced by the family

$\{U_x(\varepsilon, \lambda): x \in X, \varepsilon > 0,\ \alpha \in [0,1]$and $\lambda \in (0,1)\}$ of neighborhood.

Remark: If t is continuous, then Fuzzy F-Menger space (X, F_α^2, t) is a Hausdorff space in (ε, λ)-topology.

Let (X, F_α^2, t)be a complete Fuzzy F-Menger space and $A \subset X$. Then A is called a bounded set if

$$\lim_{u\to\infty} \inf_{x,y\in A} F_{\alpha(x,y)}^2(u) = 1$$

Definition 2.5: A sequence $\{x_n\}$ in $(X, F_\alpha^2, t)$is said to be convergent to a point x in X if for every $\varepsilon > 0$and$\lambda > 0$, there exists an integer $N = N(\varepsilon, \lambda)$ such that

$$x_n \in U_x(\varepsilon, \lambda) \quad \text{for all} \quad n \geq N$$

or equivalently $F_\alpha^2(x_n, x; \varepsilon) > 1 - \lambda$for all $n \geq N$ and $\alpha \in [0,1]$.

Definition 2.6: A sequence $\{x_n\}$ in (X, F_α^2, t) is said to be Cauchy sequence if for every $\varepsilon > 0$ and$\lambda > 0$, there exists an integer $N = N(\varepsilon, \lambda)$ such that

$$F_\alpha^2(x_n, x_m; \varepsilon) > 1 - \lambda \quad \forall\ n, m \geq N \text{ for all } \alpha \in [0,1].$$

Definition 2.7: A Fuzzy F-Menger space (X, F_α^2, t) with the continuous t-norm is said to be complete if every Cauchy sequence in X converges to a point in X for all $\alpha \in [0,1]$.

Definition 2.8: Let (X, F_α^2, t)be a Fuzzy F-Menger space. Two mappings $f, g: X \to X$ are said to be weakly compatible if they commute at coincidence point for all $\alpha \in [0,1]$.

Lemma 2.1: Let $\{x_n\}$ be a sequence in a Fuzzy F-Menger space (X, F_α^2, t), where t is continuous and $t(p,p) \geq p$ for all $p \in [0,1]$,if there exists a constant $k \in (0,1)$ such that

$$\forall\ p > 0 \text{ and } n \in N,\ t\big(F_\alpha^2(x_n, x_{n+1}; kp)\big) \geq t\big(F_\alpha^2(x_{n-1}, x_n; p)\big)$$

for all $\alpha \in [0,1]$ then $\{x_n\}$ is Cauchy sequence.

Lemma 2.2: If (X, d) is a metric space, then the metric$d$ induces, a mapping $F_\alpha: X \times X \to L$ defined by $F_\alpha^2(p,q) = H_\alpha^2(x - d(p,q))$, $p, q \in R$ for all $\alpha \in [0,1]$. Further if $t: [0,1] \times [0,1] \to [0,1]$ is defined by $t(a,b) = min\{a,b\}$, then (X, F_α^2, t) is a Fuzzy F-Menger space. It is complete if $(X, d)$is complete.

Definition 2.9: Let (X, F_α^2, t)be a Fuzzy F-Menger space. Maps $s: X \to X$ and $T: X \to CB(X)$

(1) s is said to be T weakly commuting at $x \in X\ if\ ssx \in Tsx$.

(2) are weakly compatible if the commute at their coincidence points,
i.e. if $sTx = Tsx$ whenever $sx \in Tx$.

(3) are (IT) commuting at $x \in X$ if $sTx \subset Tsx$ whenever $sx \in Tx$.

Definition 2.10: Two self maps f and g of a set X are occasionally weakly compatible iff there is a point x in X which is a coincidence point of f and g at which f and g commute.

Definition 2.11: A function $\phi: [0,\infty) \to [0,\infty)$ is said to be a$\emptyset$-function if it satisfies the following conditions:

(i) $\boldsymbol{\phi(t) = 0}$ if and only if $\boldsymbol{t = 0}$

(ii) $\boldsymbol{\phi(t)}$ is stsrictly increasing and $\boldsymbol{\phi(t)} \to \infty$ $\boldsymbol{as\ t} \to \infty$

(iii) $\boldsymbol{\phi(t)}$ is left continuous in $(\mathbf{0}, \infty)$ and

(iv) $\boldsymbol{\phi(t)}$ is continuous at **0.**

An altering distance functions with the additional property that $h(t) \to \infty$ as $t \to \infty$ generates function ϕ in the following way.

$$\phi(t) = \begin{cases} sup\{s: h(s) < t\} & if\ t > 0 \\ 0 & if\ t = 0 \end{cases}$$

It can be easily seen thatϕ is a ϕ -function.

Lemma 2.3: Let (X, F_α^2, t)be a fuzzy F-Menger space, A and B are occasionally weakly compatible self maps of X. If $A$and B have a unique point of coincidence, $w = Ax = Bx$, then w is the unique common fixed point of A and B.

Proof:- Since A and B are occasionally weakly compatible, there exists a point $x \in X$such that $Ax = Bx = w$ and $ABx = BAx$. Thus, $AAx = ABx = BAx$, which says that Ax is also a point of coincidence of A and B. Since the point of coincidence $w = Ax$ is unique by hypothesis, $BAx = AAx = Ax$,and $w = Ax$is a common fixed point of A and B. Moreover, if z is any common fixed point of A and B, then$z = Az = Bz = w$ by the uniqueness of the point of coincidence.

III. MAIN RESULTS

Theorem 3.1: Let (X, F_α^2, t) be a Menger space. Let $f, g: X \to X$ *and* $S, G: X \to CB(X)$ such that
3.1.1 (f, S) *and* (g, G) satisfy the common property (EA),
3.1.2 $f(X)$ *and* $g(X)$ are closed,
3.1.3 Pair (f, S) is S-JSR maps and pair (g, G) is G-JSR maps,
3.1.4 $t\big(H_\alpha^2(Sx, Gy, kp)\big)$
$\geq \emptyset\big[min\{\boldsymbol{t}\big(F_\alpha^2(fx, gy, p)\big), t\big(F_\alpha^2(fx, Sx, p)\big), t\big(F_\alpha^2(fx, gy, p)\big), t\big(F_\alpha^2(fx, Gy, p)\big), t\big(F_\alpha^2(Sx, gy, p)\big)\}\big]$
Then f, g, S *and* G have a common fixed point in X.

Proof: By 3.1.1 there exist two sequences $\{x_n\}$ *and* $\{y_n\}$ in X and $u \in X$, A, B *in* $CB(X)$ such that $\lim_{n\to\infty} Sx_n = A$ and $\lim_{n\to\infty} Gy_n = B$, and $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gy_n = u \in A \cap B$.
Since $f(X)$ *and* $g(X)$ are closed, we have $u = fv$ and $u = gr$ for some $v, r \in X$.
Now by 3.1.4 we get

$$t\big(F_\alpha^2(Hx_n, Gr, kp)\big)$$
$$\geq \emptyset\big[min\{t\big(F_\alpha^2(fx_n, gr, p)\big), t\big(F_\alpha^2(fx_n, Sx_n, p)\big), t\big(F_\alpha^2(gr, Gr, p)\big), t\big(F_\alpha^2(fx_n, Gr, p)\big), t\big(F_\alpha^2(Sx_n, gr, p)\big)\}\big]$$

On taking limit $n \to \infty$, we obtain

$$t\big(H_\alpha^2(A, Gr, kp)\big)$$
$$\geq \emptyset\big[min\{t\big(F_\alpha^2(fv, gr, p)\big), t\big(F_\alpha^2(fv, A, p)\big), t\big(F_\alpha^2(gr, Gr, p)\big), t\big(F_\alpha^2(fv, Gr, p)\big), t\big(F_\alpha^2(A, gr, p)\big)\}\big]$$
$$\geq \emptyset t\Big(\big(F_\alpha^2(gr, Gr, p)\big)\Big)$$
$$> t\big(F_\alpha^2(gr, Gr, p)\big) > F_\alpha^2(gr, Gr, p)$$

Since $gr = fv \in A$ and $F_\alpha^2(gr, Gr, p) \geq H_\alpha^2(A, Gr, kp) > F_\alpha^2(gr, Gr, p)$.
Hence $gr \in Gr$. Similarly

$$t\big(H_\alpha^2(Sv, Gy_n, kp)\big) \geq \emptyset\left[min\left\{\begin{matrix} t\big(F_\alpha^2(fv, gy_n, p)\big), t\big(F_\alpha^2(fv, Sv, p)\big), \\ t\big(F_\alpha^2(gy_n, Gy_n, p)\big), t\big(F_\alpha^2(fv, Gy_n, p)\big), t\big(F_\alpha^2(Sv, gy_n, p)\big)\end{matrix}\right\}\right]$$
$$t\big(H_\alpha^2(Sv, B, kp)\big) \geq \emptyset\left[min\left\{\begin{matrix} t\big(F_\alpha^2(fv, gr, p)\big), t\big(F_\alpha^2(fv, Sv, p)\big), \\ t\big(F_\alpha^2(gr, B, p)\big), t\big(F_\alpha^2(fv, B, p)\big), t\big(F_\alpha^2(Sv, gr, p)\big)\end{matrix}\right\}\right]$$
$$\geq \emptyset\Big(t\big(F_\alpha^2(fv, Sv, p)\big)\Big)$$
$$> t\big(F_\alpha^2(fv, Sv, p)\big) > F_\alpha^2(fv, Sv, p)$$

Since $fv = gr \in A$ and
$t(F_\alpha^2(fv, B, p)) \geq t(F_\alpha^2(fv, Sv, p)) > t(F_\alpha^2(fv, Sv, p))$
We get $fv \in Sv$.
Now as pair (f, S) is S-JSR maps therefore $fp \in Sp$
and similarly as pair (g, G) is G-JSR maps therefore $gu \in Gu$

$$t\big(F_\alpha^2(fx_n, gu, p)\big) \geq t\big(H_\alpha^2(Sx_n, Gu, p)\big)$$
$$\geq \emptyset\left[min\left\{\begin{matrix} t\big(F_\alpha^2(fx_n, Gu, p)\big), t\big(F_\alpha^2(fx_n, Sx_n, p)\big), \\ t\big(F_\alpha^2(gu, Gu, p)\big), t\big(F_\alpha^2(fx_n, Gu, p)\big), t\big(F_\alpha^2(Sx_n, gu, p)\big)\end{matrix}\right\}\right]$$

On taking limit $n \to \infty$, we obtain

$$t\big(F_\alpha^2(u, gu, p)\big) \geq \emptyset\left[min\left\{\begin{matrix} t\big(F_\alpha^2(u, gu, p)\big), t\big(F_\alpha^2(u, A, p)\big), \\ t\big(F_\alpha^2(gu, Gu, p)\big), t\big(F_\alpha^2(u, Gu, p)\big), t\big(F_\alpha^2(A, gu, p)\big)\end{matrix}\right\}\right]$$
$$\geq \emptyset\left[min\left\{\begin{matrix} t\big(F_\alpha^2(u, gu, p)\big), t\big(F_\alpha^2(u, A, p)\big), \\ t\big(F_\alpha^2(gu, Gu, p)\big), t\big(F_\alpha^2(u, Gu, p)\big), t\big(F_\alpha^2(A, u, p/2)\big), t\big(F_\alpha^2(u, gu, p/2)\big)\end{matrix}\right\}\right]$$

By triangular inequality and as $u \in A \cap B$, we obtain
$t(F_\alpha^2(u, Gu, p)) \geq t(F_\alpha^2(u, Gu, p)) \Longrightarrow gu = u.$
Again
$t(F_\alpha^2(fu, gx_n, p)) \geq t(H_\alpha^2(Su, Gx_n, p))$
$$\geq \emptyset \left[min \left\{ \begin{matrix} t(F_\alpha^2(fu, gx_n, p)), t(F_\alpha^2(fu, Su, p)), \\ t(F_\alpha^2(gx_n, Gx_n, p)), t(F_\alpha^2(fu, Gx_n, p)), t(F_\alpha^2(Su, u, p)) \end{matrix} \right\} \right]$$
On taking limit $n \to \infty$, we obtain
$$t(F_\alpha^2(fu, u, p)) \geq \emptyset \left[min \left\{ \begin{matrix} t(F_\alpha^2(fu, u, p)), t(F_\alpha^2(fu, Su, p)), \\ t(F_\alpha^2(u, Gu, p)), t(F_\alpha^2(fu, B, p)), t(F_\alpha^2(Su, u, p)) \end{matrix} \right\} \right]$$
$$t(F_\alpha^2(fu, u, p)) \geq \emptyset \left[min \left\{ \begin{matrix} t(F_\alpha^2(fu, u, p)), t(F_\alpha^2(fu, Su, p)), \\ t(F_\alpha^2(u, Gu, p)), t(F_\alpha^2(fu, u, p/2)) + t\left(F_\alpha^2\left(Su, u, \frac{p}{2}\right)\right), t(F_\alpha^2(Su, u, p)) \end{matrix} \right\} \right]$$
By triangular inequality and as $u \in A \cap B$, we obtain
$F_\alpha^2(fu, u, p) \geq F_\alpha^2(fu, u, p)$
$\Longrightarrow fu = u.$
Hence $u = fu \in Su \; and \; u = gu \in Su.$

Example: Let $X = [1, \infty)$ with usual metric. Define $S: X \to X \; as \; Sx = 2 + x/3$ and $T: CB(X) \to X \; as \; Tx = [1, 2 + x]$. Consider the sequence $\{x_n\} = \{3 + 1/n\}$. Then all conditions are satisfies of the theorem and hence 3 is the common fixed point for all α∈[0,1].

Theorem 3.2: (X, F_α^2, t) be a complete F Menger space, where t is continuous and $t(p, p) \geq p$ for all p in [0,1]. Let $f, g: X \to X$ and S_i, G_j are sequences of functions from X into CB(X) such that
3.2.1 $(f, S_i) \; and \; (g, G_j)$ satisfy the common property (EA),
3.2.2 $f(X) \; and \; g(X)$ are closed,
3.2.3 pair (f, S_i) is S_i-JSR maps and pair (g, G_j) is G_j-JSR maps,
3.2.4 $t(H_\alpha^2(S_i x, G_j y, kp)) \geq \emptyset \left[in \left\{ \begin{matrix} t(F_\alpha^2(fx, gy, p)), t(F_\alpha^2(fx, S_i x, p)), \\ t\left(F_\alpha^2(gy, G_j y, p)\right), t\left(F_\alpha^2(fx, G_j y, p)\right) + t(F_\alpha^2(S_i x, gy, p)), \end{matrix} \right\} \right]$
Then f, g, S_i and G_j have a common fixed point in X.
Proof: Same as above theorem for each sequences S_i and G_j.

Theorem 3.3: Let (X, F_α^2, t) be a complete F-Menger space, where t is continuous and $t(p, p) \geq p$ for all $p \; in \; [0,1]$. Let $A, B, S \; and \; T$ be self mappings from X into itself such that
3.3.1 $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$;
3.3.2 the pair (A, S) is semi compatible and (B, T) is weak compatible;
3.3.3 one of $A \; or \; S$ is continuous; for some $\phi \in \Phi$, there exist $k \in (0,1)$ such that for all $x, y \in X \; and \; p > 0$
3.3.4 $\emptyset(t(F^2(Ax, By, kp)), t(F^2(Sx, Ty, p)), t(F^2(Ax, Sx, p)), t(F^2(By, Ty, kp)) \geq 0;$
3.3.5 $\emptyset(t(F^2(Ax, By, kp)), t(F^2(Sx, Ty, p)), t(F^2(Ax, Sx, p)), t(F^2(By, Ty, kp)) \geq 0$ then $A, B, S \; and \; T$ have unique common fixed point in X.

Proof: Let x_0 be any arbitrary point of $X, as \; A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$ there exists x_1, x_2 in X such that $Ax_0 = Tx_1, Bx_1 = Sx_2$. Inductively, construct sequences $\{y_n\} \; and \; \{x_n\} \; in \; X$ such that $y_{2n+1} = Ax_{2n} = Tx_{2n+1}$, $y_{2n+2} = Bx_{2n+1} = Sx_{2n+2}$ for $n = 0,1,2, \ldots ..$
Now by (3.3.4)
$\emptyset(t(F^2(Ax_{2n}, Bx_{2n+1}, kp)), t(F^2(Sx_{2n}, Tx_{2n+1}, p)), t(F^2(Ax_{2n}, Sx_{2n}, p)), t(F^2(Ax_{2n}, Bx_{2n+1}, kp)) \geq 0$
$\Longrightarrow \emptyset(t(F^2(y_{2n+1}, y_{2n+2}, kp)), t(F^2(y_{2n}, y_{2n+1}, p), t(F^2(y_{2n+1}, y_{2n}, p), t(F^2(y_{2n+2}, y_{2n+1}, kp)) \geq 0$
$t(F^2(y_{2n+2}, y_{2n+1}, kp)) \geq t(F^2(y_{2n+1}, y_{2n}, p))$
$\Longrightarrow t(F^2(y_{2n+2}, y_{2n+1}, kp)) \geq t(F^2(y_{2n+1}, y_{2n}, p))$
Again putting $x = x_{2n+2}$ and $y = x_{2n+1}$ in (3.3.5), we have
$\emptyset(t(F^2(Ax_{2n+3}, Bx_{2n+2}, kp)), t(F^2(y_{2n+1}, y_{2n+2}, p), t\left(F^2(y_{2n+3}, y_{2n+2}, p), t(F^2(y_{2n+1}, y_{2n+2}, p))\right) \geq 0$
$F^2(y_{2n+3}, y_{2n+2}, p) \geq F^2(y_{2n+2}, y_{2n+1}, p)$
Hence by Lemma 2.1, $\{y_n\}$ is Cauchy sequence in X. Therefore $\{y_n\}$ converge to $u \; in \; X$. Therefore its subsequences $\{Ax_{2n}\}, \{Tx_{2n+1}\}, \{Bx_{2n+1}\}, \{Sx_{2n+2}\}$ also converge to u.

Case I: If S is continuous, we have

$$SAx_{2n} \to Su\ ,\ SSx_{2n} \to\ Su$$

So, weak compatibility of the pair $(A,S)\ gives\ ASx_{2n} \to\ Su\ as\ n\ \to \infty$ **Step (I):** By putting $x\ =\ Sx_{2n}\ , y\ =\ x_{2n+1}$ in (3.3.4), we obtain that

$\emptyset[t\{F^2(ASx_{2n}, Bx_{2n+1}, kp)\}, t\{F^2(SSx_{2n}, Tx_{2n+1}, p)\}, t\{F^2(ASx_{2n}, SSx_{2n}, p)\}, t\{F^2(Bx_{2n+1}, Tx_{2n+1}, kp)\}] \geq 0$

Now letting $n\ \to \infty$ and by the continuity of the t- norm, we have

$\emptyset[t\{F^2(Su, u, kp)\}, t\{F^2(Su, u, p)\}, t\{F^2(Su, Su, p)\}, t\{F^2(u, u, p)\}] \geq 0$

$\Longrightarrow \{F^2(Su, u, kp)\}, \{F^2(Su, u, p), 1,1\} \geq 0$

Now as ϕ is non-decreasing in the first argument, we have

$\Longrightarrow \phi\{F^2(Su, u, p)\}, \{F^2(Su, u, p), 1,1\} \geq 0$

Using (Fu), we get $F^2(Su, u, p) \geq\ 1$, for all $p > 0$, which gives $F^2(Su, u, p) =\ 1$, that is, $Su\ =\ u$

Step (II): By putting $x\ =\ u\ and\ y\ =\ x_{2n+1}$, we obtain that

$\emptyset[t\{F^2(Au, Bx_{2n+1}, kp)\}, t\{F^2(Su, Tx_{2n+1}, p)\}, t\{F^2(Au, Su, p)\}, t\{F^2(Bx_{2n+1}, Tx_{2n+1}, kp)\}] \geq 0$

On taking limit $n\ \to \infty\ and\ as\ Su\ =\ u\ \&\ Bx_{2n+1}, T\ x_{2n+1}\ \to\ u$, we get

$\emptyset\{F^2(Au, u, kp), 1, F^2(Au, Tu, p), 1\} \geq 0$

Now as ϕ is non-decreasing in the first argument, we have

$\emptyset\{F^2(Au, u, p), 1, F^2(Au, u, p), 1\} \geq 0$

Using , we get $F^2(Au, u, p) \geq 1$, for all $p > 0$, which gives $F^2(Au, u, p) = 1$,
that is $Au\ =\ u\ =\ Su.$

Step (III): By (3.3.1)$A(X) \subseteq T(X)$, there exists $w\ in\ X$ such that $Au =\ u = Su\ = Tw.$
By putting $x\ =\ x_{2n}$ and $y\ =\ w$ in (3.3.4) of the theorem, we obtain that

$\emptyset[t\{F^2(Ax_{2n}, Bw, kp)\}, t\{F^2(Sx_{2n}, Tw, p)\}, t\{F^2(Ax_{2n}, Sx_{2n}, p)\}, t\{F^2(Bw, Tw, kp)\}] \geq 0$

On taking limit $n\ \to \infty$ and as $Ax_{2n}, Sx_{2n} \to\ u$, we get

$\emptyset\{F^2(Au, Bw, kp), 1,1, F^2(Bw, u, kp), 1\} \geq 0$

By using, we get $F^2(u, Bw, kp) \geq 1$, for all $p\ >\ 0$, which gives $F^2(u, Bw, kp) = 1$, that is $Bw\ =\ u.$ Therefore $Bw\ =\ Tw\ =\ u.$ Since (B,T) is weak compatible, we get $TBw\ =\ BTw$, it implies $Bu\ =\ Tu.$

Step (IV): Now putting $x\ =\ u$ and $y\ =\ u$ in (3.2.4) and as $Au =\ u\ =\ Su\ \&\ Bu\ =\ Tu,$
We get,

$\emptyset[t\{F^2(Au, Bu, kp)\}, t\{F^2(Su, Tu, p)\}, t\{F^2(Au, Su, p)\}, t\{F^2(Bu, Tu, kp)\}] \geq 0$

$\emptyset[t\{F^2(Au, Bu, kp)\}, t\{F^2(Su, Tu, p)\}, 1,1] \geq 0$

Now as ϕ is non-decreasing in the first argument, we have

$\Longrightarrow \emptyset\{F^2(Au, Bu, p), F^2(Au, Bu, p), 1,1\} \geq 0$

Using (Fu), we get $F^2(Au, Bu, p) \geq 1$, for all $p > 0$, which gives $F^2(Au, Bu, p) = 1$, that is, $Au\ =\ Bu. Thus\ u\ =\ Au\ =\ Su\ =\ Bu\ =\ Tu.$

Case II: If A is continuous i.e. $ASx_{2n} \to Au.$ also the pair (A,S) is semi-compatible, therefore $ASx_{2n} \to Su$. By the uniqueness of the limit $Au\ =\ Su.$

Step (V) By putting $x\ =\ u\ and\ y\ =\ x_{2n+1}$ in (3.3.4), we get

$\emptyset[t\{F^2(Au, Bx_{2n+1}, kp)\}, t\{F^2(Su, Tx_{2n+1}, p)\}, t\{F^2(Au, Su, p)\}, t\{F^2(Bx_{2n+1}, Tx_{2n+1}, kp)\}] \geq 0$

On taking limit $n\ \to \infty$ and as $Bx_{2n+1}, Tx_{2n+1} \to u$, we get

$t\{F^2(Au, u, kp), 1, F^2(Au, u, p)\} \geq 0.$

Now as $\emptyset$ is non-decreasing in the first argument, we have

$\emptyset\{F^2(Au, u, p), 1, F^2(Au, u, p)\} \geq 0.$

Using (F_h), we have $F^2(Au, u, p) \geq 1$ for all $p > 0$, which gives $u\ = Au.$
The rest of the proof follows from step (III) of the case I.

Uniqueness of common fixed point

Let v be another common fixed point of $A, S, B\ and\ T$, then $v\ =\ Av\ =\ Sv\ =\ Bv\ =\ Tv.$ Now putting $x\ =\ u\ and\ y\ =\ v$ in (IV), we get

$\phi[t\{F^2(Au, Bv, kp)\}, t\{F^2(Su, Tv, p)\}, t\{F^2(Au, Su, p)\}, t\{F^2(Bv, Tv, kp)\}]\ \geq 0$

$\Longrightarrow \emptyset[t\{F^2(u, v, kp)\}, t\{F^2(u, v, p)\}, t\{F^2(u, u, p)\}, t\{F^2(v, v, kp)\}]\ \geq\ 0$

$\Longrightarrow \emptyset[t\{F^2(u, v, kp)\}, t\{F^2(u, v, p)\}, 1, 1]\ \geq\ 0$

Now as $\emptyset$ is non-decreasing in the first argument, we have

$\emptyset[\{F^2(u, v, p)\}, \{F^2(u, v, p)\}, 1, 1]\ \geq 0$

we have $F^2(u, v, p) \geq\ 1$ for all $p\ > 0$, which gives $u\ =\ v.$

Corollary 3.4 Let (X, F_α^2, t) be a complete F-Menger space, where t is continuous and $t(p,p) \geq p$ for all a in [0,1]. Let A, B, S and T be self mappings from X into itself such that

3.4.1 $A(X) \subseteq T(X) \cap S(X)$;

3.4.2 the pair (A, S) is semi compatible and (A, T) is weak compatible;

3.4.3 one of A or S is continuous; for some $\emptyset \in \Phi$, there exist $k \in (0,1)$ such that for all $x, y \in X$ and $p > 0$.

3.4.4 $\emptyset[t\{F^2(Ax, Ay, kp)\}, t\{F^2(Sx, Ty, p)\}, t\{F^2(Ax, Sx, p)\}, t\{F^2(Ay, Ty, kp)\}] \geq 0$;

3.4.5 $\emptyset[t\{F^2(Ax, Ay, kp)\}, t\{F^2(Sx, Ty, p)\}, t\{F^2(Ax, Sx, kp)\}, t\{F^2(Ay, Ty, p)\}] \geq 0$

then A, S and T have unique common fixed point in X.

Proof: Putting $B = A$ in **Theorem 3.3.**

Corollary 3.5 : Let (X, F_α^2, t) be a complete F-Menger space, where t is continuous and t(p,p) ≥ p for all a in [0,1]. Let A, B, S and T be self mappings from M into itself such that

3.5.1 $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$;

3.5.2. the pairs (A, S) and (A, T) are semi-compatible;

3.5.3. one of A, B, T or S is continuous; for some $\emptyset \in \Phi$, there exist $k \in (0,1)$ such that for all $x, y \in M$ and $p > 0$.

3.5.4. $\emptyset[t\{F^2(Ax, By, kp)\}, t\{F^2(Sx, Ty, p)\}, t\{F^2(Ax, Sx, p)\}, t\{F^2(By, Ty, kp)\}] \geq 0$;

3.5.5. $\phi[t\{F^2(Ax, By, kp)\}, t\{F^2(Sx, Ty, p)\}, t\{F^2(Ax, Sx, kp)\}, t\{F^2(By, Ty, p))\}] \geq 0$,

then A, B, S and T have unique common fixed point in X.

Proof: As semi-compatible mappings are weak compatible, the proof follows from Theorem 3.3.

Example 1: Let $X = [0,1]$ and metric d is defined by $d(x, y) = |x - y|$. For each p define

$$F^2(x, y, p) = \begin{cases} 1 & for \quad x = y \\ H^2(p) & x \neq y \end{cases} \text{ where } H^2(p) = \begin{cases} 0 & if \quad p \leq 0 \\ p^2 & if\ 0 < p < 1 \\ 1 & if \quad p \geq 1 \end{cases}$$

Clearly, (X, F_α^2, t) is a complete F-Menger space where t is defined by $t(p,p) \geq p$. The sequence $x_n = \frac{1}{n}$. Let A, B, S and T are defined as

$$Ax = \frac{x}{6}, Tx = x, \ Bx = x/5 \ and\ Sx = x/2. \text{ If } k = 1\ and\ p = 1.$$

So, we see the all conditions of theorem 5.3.3 are satisfied and hence 0 is the common fixed point in X.

Example 2: Let $X = [0,2]$ and metric d is defined by $d(x, y) = |x - y|$.

For each $p > 0$, we define $F^2(x, y, p) = \begin{cases} \frac{p^2}{p^2 + d(x,y)} & if \quad p > 0 \\ 0 \quad if & \quad p = 0 \end{cases}$

Define self maps A, S, B and T as follows

$$Sx = \begin{cases} \frac{1}{2} & 0 \leq x \leq 1 \\ x & 1 \leq x \leq 2 \end{cases}, Ax = \frac{x+4}{5}, \ Bx = \frac{1+x}{2} \text{ and}$$

$$Tx = \begin{cases} 1, & 0 \leq x \leq 1 \\ \frac{3-x}{2}, & 1 \leq x \leq 2 \end{cases}.$$ The sequence $\{x_n\}$ is defined as $x_n = 1 - \frac{1}{2n}$.

$B_1 = 1$ and $T_1 = 1 \Longrightarrow TB_1 = BT_1$, clearly $\{B, T\}$ is weak compatible.

$Sx_n = 1 - \dfrac{1}{2n}$ and $Ax_n = 1 - \dfrac{1}{10n}$, clearly $Ax_n \to 1$ and $Sx_n \to 1$ $i.e.$ $u = 1$.

$ASx_n = 1 - \dfrac{1}{20n}, SAx_n = \dfrac{1}{2}$. Now $\lim_{n\to\infty} F(ASx_n, Su, p) = F(1,1,p) = 1$.

Hence $\{A, S\}$ is semi compatible but not compatible as

$$\lim_{n\to\infty} F_\alpha^2(Ax_n, BAx_n, p) = \lim_{n\to\infty} F_\alpha^2\left(1 - \frac{1}{20n}, \frac{1}{2}, p\right) = \frac{p}{p + \frac{1}{2}} < 1, \forall p$$

So, for all $k \in (0,1)$ we see the all conditions of theorem 5.3.3 are satisfied and hence 1 is the common fixed point in X.

Example 3: Let $X = [0,2]$ and metric d is defined by $d(x,y) = \dfrac{|x-y|}{1+|x-y|}$. For each p

Define $F^2(x,y,p) = \begin{cases} 1 & for \quad x=y \\ H(p) & for\ x \neq y \end{cases}$, (X, F^2_α, t)

Where $H^2(p) = \begin{cases} 0 & if \quad p \leq 0 \\ p^2 d(x,y) & if \quad 0 < p < 1 \\ 1 & if \quad p \geq 1 \end{cases}$

Clearly, (X, F^2_α, t) is a complete F-Menger space where t is defined by $t(p,p) \geq p$.

$Ax = \begin{cases} 1 & 0 \leq x \leq 1 \\ \frac{4-x}{5} & 1 < x < 2 \end{cases}$, $Sx = \begin{cases} 1 & x = 1 \\ \frac{x+3}{5} & otherwise \end{cases}$,

$Bx = \begin{cases} \frac{x}{2} & 0 \leq x \leq 1/2 \\ 1 & x \geq 1/2 \end{cases}$ and $Tx = \begin{cases} 1 & 0 \leq x \leq 1 \\ \frac{x}{2} & x \geq 1/2 \end{cases}$

The sequence $\{x_n\}$ is defined as $x_n = 2 - \dfrac{1}{2n}$.

$B_1 = 1\ and\ T_1 = 1 \Longrightarrow TB_1 = BT_1\ and\ B_2 = T_2 = 1 \Longrightarrow TB_2 = BT_2$. Clearly $\{B,T\}$ is weak compatible. $Sx_n = 1 - \frac{1}{10n}$ and $Ax_n = 1 + \frac{1}{4n}$, clearly $Ax_n \to 1\ and\ Sx_n \to 1$. That is $u = 1, ASx_n = 1, SAx_n = \frac{4}{5} + \frac{1}{20n}$.

Now $\lim F^2_\alpha(ASx_n, Su, p) = F^2_\alpha(1,1,p) = 1$.

Hence $\{A,S\}$ is semi compatible but not compatible as

$\lim F^2(ASx_n, SAx_n, p) = \lim F^2(1, \frac{4}{5} + \frac{1}{20n}, p) = p\ 1/6 < 1$.

So, for all $k \in (0,1)$ we see the all conditions of Theorem 3.3 are satisfied and hence 1 is the common fixed point in X.

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